PB-groupoids vs VB-groupoids

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joint work with Alfonso Garmendia

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Exploring new arrows in the BGW-groupoid

Bielefeld, 26th October 2024

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vector bundle of rank k $\rightsquigarrow$ $\mathrm{Fr}(E) \to M$
principal $\mathrm{GL}(k)$-bundle

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$V \in \mathrm{Rep}(G)$

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What about replacing M with a Lie groupoid $\mathcal{G} \rightrightarrows M$?

VB-groupoid (Pradines, 1988) = vector bundle object in the category of Lie groupoids = Lie groupoid object in the category of vector bundles

$$
\begin{array}{ccc}
E_{\mathcal{G}} & \xrightarrow{\pi_{\mathcal{G}}} & \mathcal{G} \\
\tilde{t} \downarrow\downarrow \tilde{s} & & t \downarrow\downarrow s \\
E_M & \xrightarrow{\pi_M} & M
\end{array}
$$

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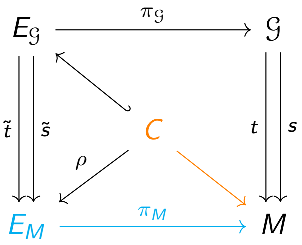

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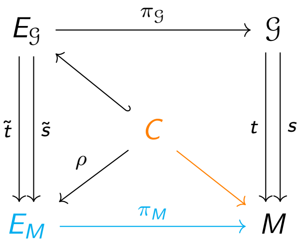

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rank $(l, k) := \left(\mathrm{rank}(C \to M), \mathrm{rank}(E_M \to M)\right)$
(so that $\mathrm{rank}(E_{\mathcal{G}}) = l + k$)

Example (tangent VB-groupoid)

$$\begin{array}{ccc} T\mathcal{G} & \xrightarrow[\pi_{\mathcal{G}}]{} & \mathcal{G} \\ dt \downarrow\downarrow ds & & t \downarrow\downarrow s \\ TM & \xrightarrow{\pi_M} & M \end{array}$$

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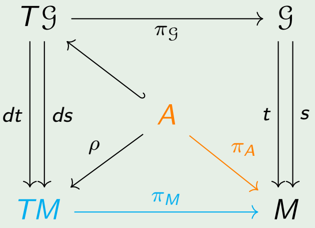

rank $= (\text{rank}(A), \dim(M))$

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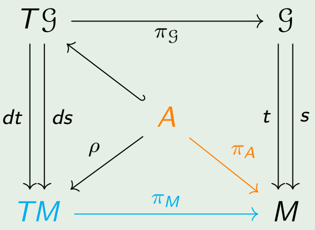

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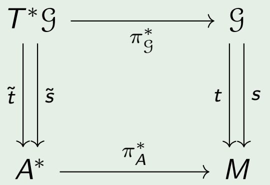

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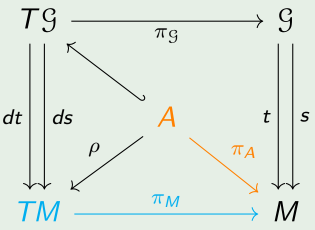

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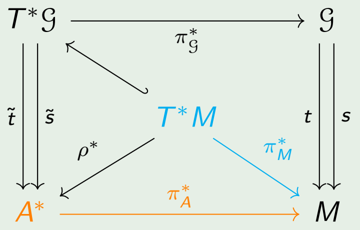

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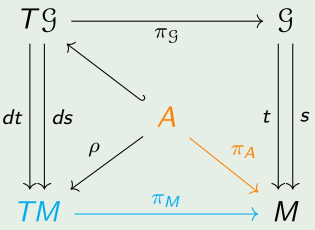

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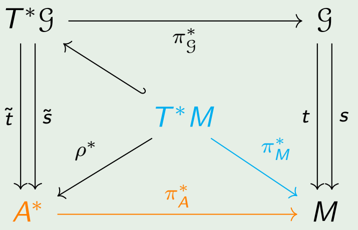

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Remark (duality between VB-groupoids)

$E_{\mathcal{G}} \rightrightarrows E_M$ VB-groupoid of rank (l, k) with core C

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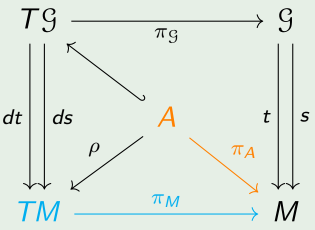

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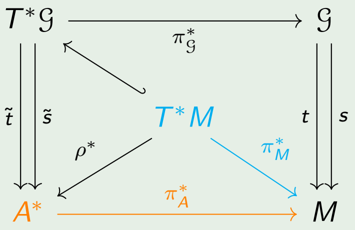

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Remark (duality between VB-groupoids)

$E_{\mathcal{G}} \rightrightarrows E_M$ VB-groupoid of rank (l, k) with core C

$\Rightarrow E^*_{\mathcal{G}} \rightrightarrows C^*$ VB-groupoid of rank (k, l) with core E^*_M

Idea: Isolate the frames of $E_{\mathcal{G}} \to \mathcal{G}$ which are compatible with the groupoid structure of $E_{\mathcal{G}} \rightrightarrows E_M$

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$$\mathbb{R}^k \xrightarrow[s(\phi_g)^b]{\cong} (E_M)_{s(g)}, \qquad \mathbb{R}^k \xrightarrow{\phi_g|_{\{0\}\times\mathbb{R}^k}} (E_{\mathcal{G}})_g \xrightarrow{\tilde{s}} (E_M)_{s(g)}$$

$$\mathbb{R}^l \xrightarrow[t(\phi_g)^c]{\cong} C_{t(g)}, \qquad \mathbb{R}^l \xrightarrow{\phi_g|_{\mathbb{R}^l\times\{0\}}} (E_{\mathcal{G}})_g \xrightarrow{\tilde{m}(\cdot, 0_{g^{-1}})} C_{t(g)}$$

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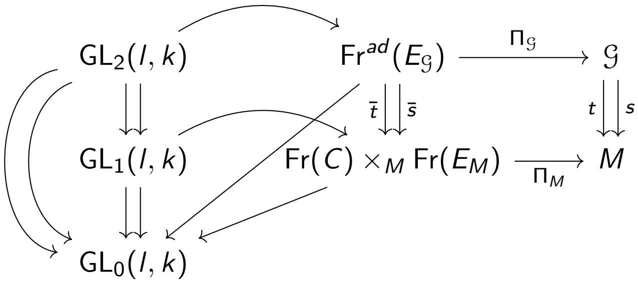
Proposition (C. - Garmendia '24)

$\mathrm{Fr}^{ad}(E_{\mathcal{G}}) \rightrightarrows \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M)$ *is a Lie groupoid.*

Proposition (C. - Garmendia '24)

There is a natural action of strict Lie 2-groupoid $\mathrm{GL}_2(l,k) \rightrightarrows \mathrm{GL}_1(l,k) \rightrightarrows \mathrm{GL}_0(l,k)$ *on the Lie groupoid* $\mathrm{Fr}^{ad}(E_{\mathcal{G}}) \rightrightarrows \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M)$



(strict) Lie 2-groupoid = double Lie groupoid over the unit groupoid

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Example (General linear 2-groupoid of rank (l, k))

- $GL_0(l, k) = \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k)$
- $GL_1(l, k) = \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k) \times GL(l) \times GL(k)$
- $GL_2(l, k) \subseteq \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k) \times GL(l + k)$

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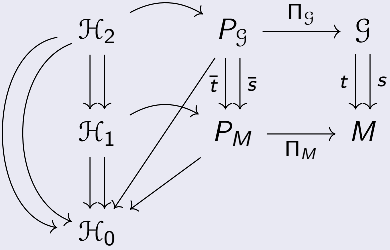
It is a particular case, for $E = (\mathbb{R}^l, \mathbb{R}^k) \to \{*\}$, of the general linear 2-groupoid $\mathrm{GL}(E)$ (del Hoyo and Stefani, 2019) and the 2-gauge groupoid 2-$\mathrm{Gau}(E)$ (Brahic and Ortiz, 2019) introduced to study 2-term RUTHs

Definition (C. - Garmendia '24)

PB-groupoid = diagram of Lie groupoids and principal bundles, with an action of a strict Lie 2-groupoid



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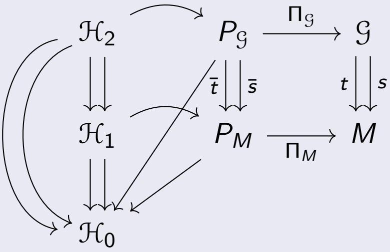

such that the action maps defines a Lie groupoid morphism $P_{\mathcal{G}} \times_{\mathcal{H}_0} \mathcal{H}_2 \to P_{\mathcal{G}}$ over $P_M \times_{\mathcal{H}_0} \mathcal{H}_1 \to P_M$

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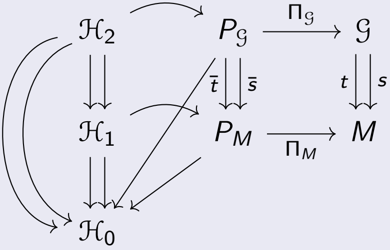

such that the action maps defines a Lie groupoid morphism $P_{\mathcal{G}} \times_{\mathcal{H}_0} \mathcal{H}_2 \to P_{\mathcal{G}}$ over $P_M \times_{\mathcal{H}_0} \mathcal{H}_1 \to P_M$, and the Lie groupoids $P_{\mathcal{G}}/\mathcal{H}_2 \rightrightarrows P_M/\mathcal{H}_1$ and $\mathcal{G} \rightrightarrows M$ are isomorphic

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$\rightsquigarrow$

$\mathrm{Fr}^{ad}(E_{\mathcal{G}}) \rightrightarrows \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M)$
PB-groupoid over $\mathcal{G} \rightrightarrows M$
with structural Lie 2-groupoid
$\mathrm{GL}(l, k)$

Proposition (C. - Garmendia '24)

$E_{\mathcal{G}} \rightrightarrows E_M$ *VB-groupoid over* $\mathcal{G} \rightrightarrows M$ *of rank* (l, k)	$\rightsquigarrow$	$\mathrm{Fr}^{ad}(E_{\mathcal{G}}) \rightrightarrows \mathrm{Fr}(C) \times_M \mathrm{Fr}(E_M)$ *PB-groupoid over* $\mathcal{G} \rightrightarrows M$ *with structural Lie 2-groupoid* $\mathrm{GL}(l, k)$
		$P_{\mathcal{G}} \rightrightarrows P_M$ *PB-groupoid over* $\mathcal{G} \rightrightarrows M$ *with structural Lie 2-groupoid* $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows X$, $E_1 \to E_0 \to X$ *2-anchored representation of* $\mathcal{H}$

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$(P_{\mathcal{G}} \times (E_1 \times_X E_0))/\mathcal{H}_2 \rightrightarrows$ $(P_M \times E_1)/\mathcal{H}_1$ *VB-groupoid over* $\mathcal{G} \rightrightarrows M$ *of rank* $(\mathrm{rank}(E_1), \mathrm{rank}(E_0))$	$\leftsquigarrow$	$P_{\mathcal{G}} \rightrightarrows P_M$ *PB-groupoid over* $\mathcal{G} \rightrightarrows M$ *with structural Lie 2-groupoid* $\mathcal{H}_2 \rightrightarrows \mathcal{H}_1 \rightrightarrows X$, $E_1 \to E_0 \to X$ *2-anchored representation of* $\mathcal{H}$

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Corollary

VB-groupoids over $\mathcal{G} \rightrightarrows M$ *of rank* (l, k)	$\longleftrightarrow$	*PB-groupoids over* $\mathcal{G} \rightrightarrows M$ *with structural Lie 2-groupoid* $\mathrm{GL}(l, k)$